Biped robot model dynamic equations

The dynamic model of the biped has the form:

$$A(q)\ddot{q} = b(q, \dot{q}, M, F), \tag{1}$$

where the generalized coordinates are collected in the vector

$$q = [x_0, y_0, \alpha, \beta_L, \beta_R, \gamma_L, \gamma_R]^T. \tag{2}$$

Vector F contains the ground support forces and M the joint moments. As the system has seven degrees of freedom, there exists also seven partial differential equations. In the following, the exact formulas of the inertia matrix $A(q)$ and the right hand side vector $b(q, \dot{q}, F, M)$ are presented.

$A(q)$:

$$A_{11} = m_0 + 2m_1 + 2m_2$$
$$A_{12} = 0$$
$$A_{13} = (-2m_1 r_0 - 2m_2 r_0)\cos(\alpha) + (-l_1 m_2 - m_1 r_1)\cos(\alpha - \beta_L) - l_1 m_2 \cos(\alpha - \beta_R) - m_1 r_1 \cos(\alpha - \beta_R) - m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) - m_2 r_2 \cos(\alpha - \beta_R + \gamma_R)$$
$$A_{14} = (l_1 m_2 + m_1 r_1)\cos(\alpha - \beta_L) + m_2 r_2 \cos(\alpha - \beta_L + \gamma_L)$$
$$A_{15} = (l_1 m_2 + m_1 r_1)\cos(\alpha - \beta_R) + m_2 r_2 \cos(\alpha - \beta_R + \gamma_R)$$
$$A_{16} = -m_2 r_2 \cos(\alpha - \beta_L + \gamma_L)$$
$$A_{17} = -m_2 r_2 \cos(\alpha - \beta_R + \gamma_R)$$
$$A_{21} = 0$$
$$A_{22} = m_0 + 2m_1 + 2m_2$$
$$A_{23} = (2m_1 r_0 + 2m_2 r_0)\sin(\alpha) + (l_1 m_2 + m_1 r_1)\sin(\alpha - \beta_L) + l_1 m_2 \sin(\alpha - \beta_R) + m_1 r_1 \sin(\alpha - \beta_R) + m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) + m_2 r_2 \sin(\alpha - \beta_R + \gamma_R)$$
$$A_{24} = (-l_1 m_2 - m_1 r_1)\sin(\alpha - \beta_L) - m_2 r_2 \sin(\alpha - \beta_L + \gamma_L)$$
$$A_{25} = (-l_1 m_2 - m_1 r_1)\sin(\alpha - \beta_R) - m_2 r_2 \sin(\alpha - \beta_R + \gamma_R)$$
$$A_{26} = m_2 r_2 \sin(\alpha - \beta_L + \gamma_L)$$
$$A_{27} = m_2 r_2 \sin(\alpha - \beta_R + \gamma_R)$$
$$A_{31} = (-2m_1 r_0 - 2m_2 r_0)\cos(\alpha) + (-l_1 m_2 - m_1 r_1)\cos(\alpha - \beta_L) - l_1 m_2 \cos(\alpha - \beta_R) - m_1 r_1 \cos(\alpha - \beta_R) - m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) - m_2 r_2 \cos(\alpha - \beta_R + \gamma_R)$$
$$A_{32} = (2m_1 r_0 + 2m_2 r_0)\sin(\alpha) + (l_1 m_2 + m_1 r_1)\sin(\alpha - \beta_L) + l_1 m_2 \sin(\alpha - \beta_R) + m_1 r_1 \sin(\alpha - \beta_R) + m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) + m_2 r_2 \sin(\alpha - \beta_R + \gamma_R)$$
$$A_{33} = 2{l_1}^2 m_2 + 2m_1 {r_0}^2 + 2m_2 {r_0}^2 + 2m_1 {r_1}^2 + 2m_2 {r_2}^2 + (2l_1 m_2 r_0 + 2m_1 r_0 r_1)\cos(\beta_L) + (2l_1 m_2 r_0 + 2m_1 r_0 r_1)\cos(\beta_R) + 2m_2 r_0 r_2 \cos(\beta_L - \gamma_L) + 2l_1 m_2 r_2 \cos(\gamma_L) + 2m_2 r_0 r_2 \cos(\beta_R - \gamma_R) + 2l_1 m_2 r_2 \cos(\gamma_R)$$

$$
\begin{aligned}
A_{34} = &- {l_1}^2 m_2 - m_1 {r_1}^2 - m_2 {r_2}^2 + (-l_1 m_2 r_0 - m_1 r_0 r_1) \cos(\beta_L) \\
&- m_2 r_0 r_2 \cos(\beta_L - \gamma_L) - 2 l_1 m_2 r_2 \cos(\gamma_L) \\
A_{35} = &- {l_1}^2 m_2 - m_1 {r_1}^2 - m_2 {r_2}^2 + (-l_1 m_2 r_0 - m_1 r_0 r_1) \cos(\beta_R) \\
&- m_2 r_0 r_2 \cos(\beta_R - \gamma_R) - 2 l_1 m_2 r_2 \cos(\gamma_R) \\
A_{36} = &\, m_2 r_2 (r_2 + r_0 \cos(\beta_L - \gamma_L) + l_1 \cos(\gamma_L)) \\
A_{37} = &\, m_2 r_2 (r_2 + r_0 \cos(\beta_R - \gamma_R) + l_1 \cos(\gamma_R)) \\
A_{41} = &\, (l_1 m_2 + m_1 r_1) \cos(\alpha - \beta_L) + m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) \\
A_{42} = &\, (-l_1 m_2 - m_1 r_1) \sin(\alpha - \beta_L) - m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) \\
A_{43} = &- {l_1}^2 m_2 - m_1 {r_1}^2 - m_2 {r_2}^2 + r_0 (-l_1 m_2 - m_1 r_1) \cos(\beta_L) \\
&- m_2 r_0 r_2 \cos(\beta_L - \gamma_L) - 2 l_1 m_2 r_2 \cos(\gamma_L) \\
A_{44} = &\, {l_1}^2 m_2 + m_1 {r_1}^2 + m_2 {r_2}^2 + 2 l_1 m_2 r_2 \cos(\gamma_L) \\
A_{45} = &\, 0 \\
A_{46} = &\, m_2 r_2 (-r_2 - l_1 \cos(\gamma_L)) \\
A_{47} = &\, 0 \\
A_{51} = &\, (l_1 m_2 + m_1 r_1) \cos(\alpha - \beta_R) + m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
A_{52} = &\, (-l_1 m_2 - m_1 r_1) \sin(\alpha - \beta_R) - m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
A_{53} = &- {l_1}^2 m_2 - m_1 {r_1}^2 - m_2 {r_2}^2 + r_0 (-l_1 m_2 - m_1 r_1) \cos(\beta_R) \\
&- m_2 r_0 r_2 \cos(\beta_R - \gamma_R) - 2 l_1 m_2 r_2 \cos(\gamma_R) \\
A_{54} = &\, 0 \\
A_{55} = &\, {l_1}^2 m_2 + m_1 {r_1}^2 + m_2 {r_2}^2 + 2 l_1 m_2 r_2 \cos(\gamma_R) \\
A_{56} = &\, 0 \\
A_{57} = &\, m_2 r_2 (-r_2 - l_1 \cos(\gamma_R)) \\
A_{61} = &- m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) \\
A_{62} = &\, m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) \\
A_{63} = &\, m_2 r_2 (r_2 + r_0 \cos(\beta_L - \gamma_L) + l_1 \cos(\gamma_L)) \\
A_{64} = &\, m_2 r_2 (-r_2 - l_1 \cos(\gamma_L)) \\
A_{65} = &\, 0 \\
A_{66} = &\, m_2 {r_2}^2 \\
A_{67} = &\, 0 \\
A_{71} = &- m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
A_{72} = &\, m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
A_{73} = &\, m_2 r_2 (r_2 + r_0 \cos(\beta_R - \gamma_R) + l_1 \cos(\gamma_R)) \\
A_{74} = &\, 0 \\
A_{75} = &\, m_2 r_2 (-r_2 - l_1 \cos(\gamma_R)) \\
A_{76} = &\, 0 \\
A_{77} = &\, m_2 {r_2}^2
\end{aligned}
$$

$b(q, \dot{q}, F, M)$:

$$
\begin{aligned}
b_1 = & -2\dot{\alpha}^2 m_1 r_0 \sin(\alpha) + F_{Rx} - \dot{\beta}_R^2 m_1 r_1 \sin(\alpha - \beta_R) - \dot{\alpha}^2 l_1 m_2 \sin(\alpha \\
& - \beta_L) + F_{Lx} - \dot{\gamma}_L^2 m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) - \dot{\alpha}^2 m_1 r_1 \sin(\alpha - \beta_R) \\
& - \dot{\beta}_R^2 l_1 m_2 \sin(\alpha - \beta_R) - \dot{\alpha}^2 l_1 m_2 \sin(\alpha - \beta_R) - \dot{\gamma}_R^2 m_2 r_2 \sin(\alpha \\
& - \beta_R + \gamma_R) - \dot{\beta}_L^2 m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) - \dot{\beta}_L^2 m_1 r_1 \sin(\alpha - \beta_L) \\
& - \dot{\beta}_R^2 m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) - \dot{\alpha}^2 m_1 r_1 \sin(\alpha - \beta_L) \\
& - \dot{\beta}_L^2 l_1 m_2 \sin(\alpha - \beta_L) - \dot{\alpha}^2 m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
& - \dot{\alpha}^2 m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) + 2\dot{\alpha}\dot{\beta}_R l_1 m_2 \sin(\alpha - \beta_R) \\
& + 2\dot{\beta}_R \dot{\gamma}_R m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) - 2\dot{\alpha}\dot{\gamma}_R m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
& + 2\dot{\alpha}\dot{\beta}_L m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) + 2\dot{\alpha}\dot{\beta}_L m_1 r_1 \sin(\alpha - \beta_L) \\
& + 2\dot{\alpha}\dot{\beta}_R m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) + 2\dot{\alpha}\dot{\beta}_L l_1 m_2 \sin(\alpha - \beta_L) \\
& - 2\dot{\alpha}^2 m_2 r_0 \sin(\alpha) + 2\dot{\alpha}\dot{\beta}_R m_1 r_1 \sin(\alpha - \beta_R) \\
& + 2\dot{\beta}_L \dot{\gamma}_L m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) - 2\dot{\alpha}\dot{\gamma}_L m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) \\
b_2 = & -\dot{\beta}_L^2 m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) + F_{Ry} - 2g m_2 - \dot{\gamma}_R^2 m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
& - \dot{\beta}_R^2 m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) - \dot{\alpha}^2 m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
& - \dot{\gamma}_L^2 m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) - \dot{\alpha}^2 m_1 r_1 \cos(\alpha - \beta_R) \\
& - \dot{\beta}_R^2 m_1 r_1 \cos(\alpha - \beta_R) - \dot{\alpha}^2 (2m_1 + 2m_2) r_0 \cos(\alpha) - \dot{\alpha}^2 l_1 m_2 \cos(\alpha \\
& - \beta_R) - \dot{\beta}_R^2 l_1 m_2 \cos(\alpha - \beta_R) + F_{Ly} - \dot{\alpha}^2 m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) - 2g m_1 \\
& + 2\dot{\alpha}\dot{\beta}_R l_1 m_2 \cos(\alpha - \beta_R) + 2\dot{\beta}_R \dot{\gamma}_R m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
& - 2\dot{\alpha}\dot{\gamma}_R m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) + 2\dot{\alpha}\dot{\beta}_R m_2 r_2 \cos(\alpha - \beta_R + \gamma_R) \\
& - 2\dot{\alpha}\dot{\gamma}_L m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) + 2\dot{\beta}_L \dot{\gamma}_L m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) \\
& + 2\dot{\alpha}\dot{\beta}_L m_2 r_2 \cos(\alpha - \beta_L + \gamma_L) + 2\dot{\alpha}\dot{\beta}_R m_1 r_1 \cos(\alpha - \beta_R) \\
& - (\dot{\alpha}\dot{\beta}_L(-2l_1 m_2 - 2m_1 r_1) + \dot{\alpha}^2 (l_1 m_2 + m_1 r_1) + \dot{\beta}_L^2 (l_1 m_2 \\
& + m_1 r_1)) \cos(\alpha - \beta_L) - g m_0
\end{aligned}
$$

$$
\begin{aligned}
b_3 = {} & \dot\gamma_R^2 l_1 m_2 r_2 \sin(\gamma_R) + F_{Ry} l_2 \sin(\alpha - \beta_R + \gamma_R) - F_{Lx} l_1 \cos(\alpha - \beta_L) \\
& - F_{Rx} l_1 \cos(\alpha - \beta_R) - F_{Lx} l_2 \cos(\alpha - \beta_L + \gamma_L) + F_{Ly} r_0 \sin(\alpha) + F_{Ry} r_0 \sin(\alpha) \\
& + F_{Ly} l_1 \sin(\alpha - \beta_L) + F_{Ly} l_2 \sin(\alpha - \beta_L + \gamma_L) - g m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
& - \dot\gamma_R^2 m_2 r_0 r_2 \sin(\beta_R - \gamma_R) - \dot\beta_R^2 m_2 r_0 r_2 \sin(\beta_R - \gamma_R) \\
& - g m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) + \dot\gamma_L^2 l_1 m_2 r_2 \sin(\gamma_L) \\
& - \dot\gamma_L^2 m_2 r_0 r_2 \sin(\beta_L - \gamma_L) - \dot\beta_R^2 m_1 r_0 r_1 \sin(\beta_R) \\
& - \dot\beta_L^2 m_2 r_0 r_2 \sin(\beta_L - \gamma_L) - \dot\beta_R^2 l_1 m_2 r_0 \sin(\beta_R) \\
& - \dot\beta_L^2 m_1 r_0 r_1 \sin(\beta_L) - g l_1 m_2 \sin(\alpha - \beta_R) + F_{Ry} l_1 \sin(\alpha - \beta_R) - g m_1 r_1 \sin(\alpha \\
& - \beta_R) - g l_1 m_2 \sin(\alpha - \beta_L) - g m_1 r_1 \sin(\alpha - \beta_L) - \dot\beta_L^2 l_1 m_2 r_0 \sin(\beta_L) \\
& - (F_{Lx} r_0 + F_{Rx} r_0) \cos(\alpha) - 2\dot\alpha\dot\gamma_L m_2 r_0 r_2 \sin(\beta_L - \gamma_L) \\
& + 2\dot\beta_L\dot\gamma_L m_2 r_0 r_2 \sin(\beta_L - \gamma_L) + 2\dot\alpha\dot\beta_L m_2 r_0 r_2 \sin(\beta_L \\
& - \gamma_L) + 2\dot\alpha\dot\beta_R m_1 r_0 r_1 \sin(\beta_R) - 2 g m_2 r_0 \sin(\alpha) \\
& + 2\dot\alpha\dot\gamma_R l_1 m_2 r_2 \sin(\gamma_R) + 2\dot\alpha\dot\beta_R m_2 r_0 r_2 \sin(\beta_R - \gamma_R) \\
& - 2\dot\alpha\dot\gamma_R m_2 r_0 r_2 \sin(\beta_R - \gamma_R) + 2\dot\alpha\dot\gamma_L l_1 m_2 r_2 \sin(\gamma_L) \\
& - 2\dot\beta_L\dot\gamma_L l_1 m_2 r_2 \sin(\gamma_L) - 2 g m_1 r_0 \sin(\alpha) \\
& + 2\dot\alpha\dot\beta_R l_1 m_2 r_0 \sin(\beta_R) + 2\dot\alpha\dot\beta_L l_1 m_2 r_0 \sin(\beta_L) \\
& + 2\dot\alpha\dot\beta_L m_1 r_0 r_1 \sin(\beta_L) - 2\dot\beta_R\dot\gamma_R l_1 m_2 r_2 \sin(\gamma_R) \\
& + 2\dot\beta_R\dot\gamma_R m_2 r_0 r_2 \sin(\beta_R - \gamma_R) - F_{Rx} l_2 \cos(\alpha - \beta_R + \gamma_R) \\
b_4 = {} & M_{L1} + F_{Lx} l_1 \cos(\alpha - \beta_L) + F_{Lx} l_2 \cos(\alpha - \beta_L + \gamma_L) - F_{Ly} l_1 \sin(\alpha - \beta_L) \\
& + g l_1 m_2 \sin(\alpha - \beta_L) + g m_1 r_1 \sin(\alpha - \beta_L) - \dot\alpha^2 l_1 m_2 r_0 \sin(\beta_L) \\
& - \dot\alpha^2 m_1 r_0 r_1 \sin(\beta_L) - \dot\alpha^2 m_2 r_0 r_2 \sin(\beta_L - \gamma_L) \\
& - 2\dot\alpha\dot\gamma_L l_1 m_2 r_2 \sin(\gamma_L) + 2\dot\beta_L\dot\gamma_L l_1 m_2 r_2 \sin(\gamma_L) \\
& - \dot\gamma_L^2 l_1 m_2 r_2 \sin(\gamma_L) - F_{Ly} l_2 \sin(\alpha - \beta_L + \gamma_L) + g m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) \\
b_5 = {} & M_{R1} + F_{Rx} l_1 \cos(\alpha - \beta_R) + F_{Rx} l_2 \cos(\alpha - \beta_R + \gamma_R) - F_{Ry} l_1 \sin(\alpha - \beta_R) \\
& + g l_1 m_2 \sin(\alpha - \beta_R) + g m_1 r_1 \sin(\alpha - \beta_R) - \dot\alpha^2 l_1 m_2 r_0 \sin(\beta_R) \\
& - \dot\alpha^2 m_1 r_0 r_1 \sin(\beta_R) - \dot\alpha^2 m_2 r_0 r_2 \sin(\beta_R - \gamma_R) \\
& - 2\dot\alpha\dot\gamma_R l_1 m_2 r_2 \sin(\gamma_R) + 2\dot\beta_R\dot\gamma_R l_1 m_2 r_2 \sin(\gamma_R) \\
& - \dot\gamma_R^2 l_1 m_2 r_2 \sin(\gamma_R) - F_{Ry} l_2 \sin(\alpha - \beta_R + \gamma_R) + g m_2 r_2 \sin(\alpha - \beta_R + \gamma_R) \\
b_6 = {} & M_{L2} - F_{Lx} l_2 \cos(\alpha - \beta_L + \gamma_L) + \dot\alpha^2 m_2 r_0 r_2 \sin(\beta_L - \gamma_L) \\
& - \dot\alpha^2 l_1 m_2 r_2 \sin(\gamma_L) + 2\dot\alpha\dot\beta_L l_1 m_2 r_2 \sin(\gamma_L) \\
& - \dot\beta_L^2 l_1 m_2 r_2 \sin(\gamma_L) + F_{Ly} l_2 \sin(\alpha - \beta_L + \gamma_L) - g m_2 r_2 \sin(\alpha - \beta_L + \gamma_L) \\
b_7 = {} & M_{R2} - F_{Rx} l_2 \cos(\alpha - \beta_R + \gamma_R) + \dot\alpha^2 m_2 r_0 r_2 \sin(\beta_R - \gamma_R) \\
& - \dot\alpha^2 l_1 m_2 r_2 \sin(\gamma_R) + 2\dot\alpha\dot\beta_R l_1 m_2 r_2 \sin(\gamma_R) \\
& - \dot\beta_R^2 l_1 m_2 r_2 \sin(\gamma_R) + F_{Ry} l_2 \sin(\alpha - \beta_R + \gamma_R) - g m_2 r_2 \sin(\alpha - \beta_R + \gamma_R)
\end{aligned}
$$